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NOTE

Simple Proof of Hardness of Feedback Vertex Set

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Abstract: The Feedback Vertex Set problem (FVS), where the goal is to find a small subset of vertices that intersects every cycle in an input directed graph, is among the fundamental problems whose approximability is not well understood. One can efficiently find an $\widetilde{O}(\log n)$-factor approximation, and efficient constant-factor approximation is ruled out under the Unique Games Conjecture (UGC). We give a simpler proof that Feedback Vertex Set is hard to approximate within any constant factor, assuming UGC.

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1 Introduction

Feedback Vertex Set (FVS) is a fundamental combinatorial optimization problem. Given a (directed) graph G, the problem asks to find a subset F of vertices[1] with the minimum cardinality that intersects every cycle in the graph (equivalently, the induced subgraph $G \setminus F$ is acyclic). One of Karp's 21 NP-complete problems, FVS has been a subject of active research for many years. Recent results on the

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[1]The related Feedback *Arc* Set problem asks for a subset of *edges* to intersect every cycle. This problem is easy on undirected graphs, and equivalent to FVS for directed graphs. In this paper, we deal with the vertex variant.

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problem study *approximability* and *fixed-parameter tractability*. In fixed-parameter tractability, both undirected and directed FVS are shown to be in FPT [4, 5]. See recent results on a generalization of FVS [8, 6] and references therein. In this work, we focus on approximability.

FVS in undirected graphs has a 2-approximation algorithm [1, 3, 7], but the same problem is not well understood in directed graphs The best approximation algorithm [16, 11, 10] achieves an approximation factor of $O(\min(\log \tau^* \log\log \tau^*, \log n \log\log n))$, where τ^* is the optimal fractional solution in the natural LP relaxation.[2] The best hardness result follows from an easy approximation-preserving reduction from Vertex Cover by Dinur and Safra [9], which implies that it is NP-hard to approximate FVS within a factor of 1.36. Assuming the Unique Games Conjecture (UGC) of Khot [13], it is NP-hard (called *UG-hard*) to approximate FVS in directed graphs within any constant factor [12, 17].

The first UG-hardness of approximating FVS within a constant factor is the corollary of the fairly complicated result on the Maximum Acyclic Subgraph (MAS) by Guruswami et al. [12]. Svensson [17] gave a simpler proof tailored for FVS with a stronger statement in completeness—deleting $(1+\varepsilon)/k$ fraction of vertices ensures that there is no walk of length k. The main contribution of this work is a simpler proof of the same statement. Let $[i]_k$ be the integer in $\{1,\dots k\}$ such that $i \equiv [i]_k \bmod k$.

Theorem 1.1. *Fix an integer $k \geq 3$ and $\varepsilon \in (0, 1/(k+1))$. Given a directed graph $G = (V_G, E_G)$, it is UG-hard to distinguish the following cases.*

- *Completeness: The vertex set can be partitioned into sets $V_0, \dots, V_k$ such that*

$$|V_i| \geq \frac{(1-\varepsilon)}{k}|V_G|$$

  for all $i \in \{1,\dots,k\}$, and each edge not incident on V_0 goes from V_i to $V_{[i+1]_k}$ for some $i \in \{1,\dots,k\}$.

- *Soundness: Any subset of measure ε contains a k-cycle.*

Consequently, it is UG-hard to approximate FVS within a factor of k, for any constant k.

Our proof differs from Svensson's [17] in two aspects:

- The ingenious application of *It Ain't Over Till It's Over* Theorem is replaced by the standard application of the more general *Invariance principle* of Mossel [15].
- The reduction from Unique Games is simpler, introducing only one *long code* for each vertex of a Unique Games instance, while [17] used multiple long codes for each tuple of vertices of a certain length. Instead we rely on the stronger (but equivalent) UGC proposed by Khot and Regev [14].

The idea of using the Invariance principle to prove hardness of FVS is inspired by the elegant paper of Bansal and Khot [2] which showed structured hardness of k-Hypergraph Vertex Cover. Our main idea for this result is to use a more restricted distribution for the dictatorship test than the one used in [2] to ensure more structure in the completeness case. Our statement for the completeness case is stronger that of Theorem 1.1 of Svensson [17], but his technique also proves our statement. At the same time we also ensure that the distribution has certain properties so that the same soundness analysis can be applied.

[2]In unweighted cases, τ^* is always at most n. In weighted cases, we assume all weights are at least 1.

Notation. In a directed graph $G = (V_G, E_G)$, an edge (u, v) indicates a directed edge from u to v. In some cases G might be vertex-weighted or edge-weighted, and every weight will be normalized so that the sum is 1. Given a subset S of either V_G or E_G, define $\mu(S)$, also called the *measure* of S, to be the sum of the weights of the elements in S. Let $[k] := \{1, 2, \ldots, k\}$. We often consider *hypercube* or *long code* $[k]^R$. We use superscripts $x^1, \ldots, x^k \in [k]^R$ to denote k different points of the hypercube and subscripts $x_1, \ldots, x_R$ to denote the value of each coordinate of one point $x \in [k]^R$.

Organization. In Section 2, we propose our *dictatorship test*. It is a family of instances of FVS where every small feedback vertex set must exhibit a certain structure, and the proposal and the analysis of the dictatorship test is our main technical contribution. Using the dictatorship test, Section 3 shows the full reduction from Unique Games to FVS, which is rather standard in the literature.

2 Dictatorship test

There is a simple gap-preserving reduction from FVS on vertex-weighted graphs to FVS on unweighted graphs—replace each vertex v by a set of new vertices $s(v)$ whose cardinality is proportional to the weight of v, and replace each edge (u, v) by $\{(u', v') : u' \in s(u), v' \in s(v)\}$. Our proof will have all the weights polynomially bounded, ensuring that this reduction runs in polynomial time. For the rest of the paper, we focus on vertex-weighted graphs.

We propose a simple dictatorship test for FVS, which is used to prove that it is UG-hard to approximate FVS within any constant factor. Given positive integers k, R, and $\varepsilon > 0$, our dictatorship test is a vertex-weighted graph $G = (V_G, E_G)$ where $V_G = ([k] \cup \{0\})^R$ and edges in E_G are carefully chosen to prove the following properties (informally stated).

- Completeness: For each $1 \le j \le R$, *depending only on the j-th coordinate*, V_G can be partitioned to $k+1$ parts $V_0, \ldots, V_k$ with the following two properties.
  - $\mu(V_0) = \varepsilon$, $\mu(V_1) = \cdots = \mu(V_k) = (1-\varepsilon)/k$.
  - In the subgraph induced by $V_1 \cup \cdots \cup V_k$, each edge goes from V_i to $V_{[i+1]_k}$ for some $1 \le i \le k$.

  It is easy to see that $V_0 \cup V_i$ for any $1 \le i \le k$ gives a feedback vertex set with measure
  $$\varepsilon + \frac{1-\varepsilon}{k}.$$
- Soundness: Any subset of measure at least ε that does not reveal any *influential coordinate* must contain a k-cycle.

Before defining G, we first define a k-uniform hypergraph $H = (V_H, E_H)$ with $V_H = V_G = (\{0\} \cup [k])^R$. The graph G is then simply obtained by replacing a hyperedge $(x^1, \ldots, x^k)$ by k edges $(x^1, x^2), \ldots, (x^k, x^1)$. The hypergraph H is vertex-weighted and edge-weighted. Both weights sum to 1 and induce probability distributions, where the weight of vertex x is the sum of the weight of the hyperedges containing x divided by k. The hyperedges of H are described by the following procedure to sample k vertices $(x^1, \ldots, x^k)$ from $(\{0\} \cup [k])^R$, with the weight of each hyperedge equal to the probability that it is sampled in this procedure.

- For each coordinate $1 \leq j \leq R$, sample $(x^1)_j, \ldots, (x^k)_j$ as follows, independently of the other coordinates.
  - Sample $a \in [k]$ uniformly at random.
  - Set $(x^1)_j = a, (x^2)_j = [a+1]_k, \ldots, (x^k)_j = [a+k-1]_k$.
  - For each $(x^i)_j$, set $(x^i)_j = 0$ with probability ε independently.

This defines the hypergraph E_H. In the above distribution to sample $(x^1, \ldots, x^k)$, the marginal on each x^i is the same:

$$\Pr[x^i = (a_1, \ldots, a_R)] = \prod_{j=1}^{R} \mu(a_j),$$

where $\mu : [k] \cup \{0\} \to \mathbb{R}$ is defined by $\mu(0) = \varepsilon$ and $\mu(i) = (1-\varepsilon)/k$ for $i \in [k]$. Let the weight of $(x_1, \ldots, x_R)$ be this quantity. The sum of the vertex weights is also 1.

With nonzero probability a randomly sampled hyperedge $(x^1, \ldots, x^k)$ might have $x^i = x^j$ for some $i \neq j$. We call such hyperedges *defective* since they do not make H k-uniform. However, $x^i = x^j$ means $x^i = x^j = (0, 0, \ldots, 0)$, so the probability that it happens is at most ε^{2R} and the sum of the weights of the defective hyperedges is at most $k^2 \varepsilon^{2R}$.

Finally, we define G. The vertex set $V_G = V_H$ with the same vertex weights, and for each non-defective hyperedge $(x^1, \ldots, x^k) \in E_H$, we add k edges $(x^1, x^2), \ldots, (x^k, x^1)$ to E_G. The analysis dealing with edge weights will be done in H, so we do not consider edge weights for the edges of G.

2.1 Analysis of dictatorship test

Completeness. Fix a coordinate $1 \leq j \leq R$. For all $0 \leq i \leq k$, let $V_i = \{(x_1, \ldots, x_R) \in V_G : x_j = i\}$. By definition, $\mu(V_0) = \varepsilon, \mu(V_i) = (1-\varepsilon)/k$. The distribution on $(x^1, \ldots, x^k)$ satisfies that for any $1 \leq i \leq k$, $(x^{[i+1]_k})_j = [(x^i)_j + 1]_k$ or at least one of $(x^i)_j, (x^{[i+1]_k})_j$ is 0. This proves that if we delete V_0 and the edges incident on it, all the remaining edges will go from V_i to $V_{[i+1]_k}$.

Soundness. We introduce some definitions and properties of correlated spaces and Fourier analysis of functions defined on (the products of) these spaces. See Mossel [15] for details.

Let $\Omega := [k] \cup \{0\}$ and $\mu : \Omega \to \mathbb{R}$ such that $\mu(0) = \varepsilon$ and $\mu(i) = (1-\varepsilon)/k$ as defined previously. Let (Ω^k, μ') be the probability space defined by the distribution of $(x^1)_j, \ldots, (x^k)_j$ for some j from our hyperedge sampling. Note that the marginal distribution of each copy of Ω is μ. Given a probability space $(\Omega_1 \times \Omega_2, \nu)$, we define the correlation between Ω_1 and Ω_2 as

$$\rho(\Omega_1, \Omega_2; \nu) = \sup\left\{\mathsf{Cov}[f, g] : f \in \mathbb{R}^{\Omega_1}, g \in \mathbb{R}^{\Omega_2}, \mathsf{Var}[f] = \mathsf{Var}[g] = 1\right\}.$$

With more than two underlying spaces, the correlation of $(\Omega_1 \times \cdots \times \Omega_k, \nu)$ is defined by

$$\rho(\Omega_1, \ldots, \Omega_k; \nu) = \max_{1 \leq i \leq k} \rho\left(\prod_{j=1}^{i-1} \Omega_j \times \prod_{j=i+1}^{k} \Omega_j, \Omega_i; \nu\right).$$

We use the following lemma to bound the correlation.

Lemma 2.1 (Lemma 2.9 of Mossel [15]). *Let $(\Omega_1 \times \Omega_2, \nu)$ be a probability space such that the probability of the smallest atom in $\Omega_1 \times \Omega_2$ is at least $\gamma > 0$. Define a bipartite graph $G = (\Omega_1 \cup \Omega_2, E)$ where $(a,b) \in \Omega_1 \times \Omega_2$ satisfies $(a,b) \in E$ if $\nu(a,b) > 0$. Then if G is connected then*

$$\rho(\Omega_1, \Omega_2; \nu) \leq 1 - \gamma^2/2\,.$$

In our distribution (Ω^k, μ'), note that $(0,0,\ldots,0)$ has probability $\gamma := \varepsilon^k$, and this is indeed the smallest nonzero probability assuming $\varepsilon < 1/(k+1)$. Let $\Omega_1 = \Omega$, $\Omega_2 = \Omega^{k-1}$, and consider the bipartite graph defined above. For any $(x_1,\ldots,x_k)$ with nonzero probability, the edge corresponding to $(x_1,\ldots,x_k)$ is connected to the edge corresponding to $(0,0,\ldots,0)$ since $(x_1,x_2,\ldots,x_k)$, $(0,x_2,\ldots,x_k)$, $(0,0,\ldots,0)$ is the sequence of elements with nonzero probability where each consecutive elements differ in exactly one of Ω_1 or Ω_2 (i. e., consecutive edges share an endpoint in the bipartite graph). Therefore, we can apply the above lemma to see $\rho(\Omega_1,\Omega_2;\mu') \leq 1-\gamma^2/2$. Since every copy of Ω is identical under μ', $\rho := \rho(\Omega,\ldots,\Omega;\mu') \leq 1-\gamma^2/2 < 1$. The similar argument also works for $\Omega_1 = \Omega^j$ and $\Omega_2 = \Omega^{k-j}$ for each $j \in \{1,\ldots,k-1\}$, proving that

$$\rho(\Omega^j, \Omega^{k-j}, \mu') \leq 1-\gamma^2/2\,, \quad \text{for each } j \in \{1,\ldots,k-1\}.$$

Let $\chi_0,\ldots,\chi_k \in \mathbb{R}^\Omega$ be orthonormal random variables satisfying that $\chi_0 \equiv 1$, $\mathbb{E}[\chi_i^2] = 1$ for all i, and $\mathbb{E}[\chi_i\chi_j] = 0$ for all $i \neq j$. Given $f : \Omega^R \to [0,1]$ as a random variable in the probability space $(\Omega^R, \mu^{\otimes R})$, its multilinear decomposition is

$$f(x_1,\ldots,x_R) = \sum_{\alpha \in \Omega^R} \hat{f}(\alpha) \prod_{j=1}^{R} \chi_{\alpha(j)}(x_j)\,.$$

Let $\mathsf{Supp}(\alpha)$ be the number of nonzero coordinates of α. The d-degree influence of the j-th coordinate of f is defined by

$$\mathsf{Inf}_j^{\leq d}(f) = \sum_{\alpha \in \Omega^R : \alpha_j \neq 0, \mathsf{Supp}(\alpha) \leq d} \hat{f}(\alpha)^2.$$

It is well known that $\sum_{j=1}^R \mathsf{Inf}_j^{\leq d}(f) \leq d$ for $[0,1]$-valued f and does not depend on the choice of $\chi_0,\ldots,\chi_k$.

We establish the soundness property using the Invariance principle stated below.

Theorem 2.2 (Theorem 6.3 of Mossel [15]). *Let $(\prod_{i=1}^k \Omega_i, \nu)$ be a probability space such that the minimum probability of any atom is at least $\gamma > 0$. Assume furthermore that there exists $\rho < 1$ such that*

$$\rho(\Omega_1,\ldots,\Omega_k,\nu) \leq \rho\,,$$
$$\rho\Big(\prod_{l=1}^{i} \Omega_l, \prod_{l=i+1}^{k} \Omega_l, \nu\Big) \leq \rho, \quad \textit{for all } i \in \{1,\ldots,k-1\}\,.$$

Then for all $\beta > 0$, there exist $\delta > 0, \tau > 0$, and an integer d such that the following holds. Fix a natural number R and consider the space $(\prod_{i=1}^k \Omega_i^R, \nu^{\otimes R})$. If k functions $\{f_i : \Omega_i^R \mapsto [0,1]\}_{1 \leq i \leq k}$ satisfy

$$\mathbb{E}[f_i] \geq \beta\,, \qquad i \in [k]\,,$$
$$\mathsf{Inf}_j^{\leq d}(f_i) \leq \tau\,, \qquad \forall i \in [k], j \in [R]\,,$$

then

$$\mathbb{E}\left[\prod_{i=1}^{k} f_j\right] \geq \delta.$$

The original statement of Mossel [15] is more general than the above statement and lower bounds $\mathbb{E}[\prod_{i=1}^{k} f_i]$ by some quantity $\Gamma := \Gamma(\rho,\beta)$ minus some additive error. Given $\rho < 1$ and $\beta > 0$, our statement is obtained by observing that $\Gamma(\rho,\beta) > 0$ and setting the additive error to be $\Gamma(\rho,\beta)/2$ so that $\delta := \Gamma(\rho,\beta)/2$ becomes a lower bound of $\mathbb{E}[\prod_{j=1}^{k} f_j]$. We refer the reader to [15] for the original statement.

Let A be the subset of V_G of measure at least β, and f be its indicator function. Apply Theorem 2.2 with ρ, β, and $\nu \leftarrow \mu'$ to have δ, τ and d. If $\mathsf{Inf}_j(f) \leq \tau$ for all $j \in [R]$ (i. e., A does not reveal any influential coordinate), as long as δ is greater than the sum of the weights of the defective hyperedges, which is at most $k^2\varepsilon^{2R}$ (which can be ensured by taking large R for fixed k and ε), A contains a non-defective hyperedge $(x^1,\dots,x^k)$ of H and the corresponding k-cycle of G. By taking $\beta \leftarrow \varepsilon$, we can conclude that any subset of measure at least ε that does not reveal any influential coordinate must contain a k-cycle, establishing the desired soundness property.

3 Reduction from the Unique Games

We introduce the Unique Games Conjecture and its equivalent variant.

Definition 3.1. An instance

$$\mathcal{L}\left(B(V_B \cup W_B, E_B), [R], \{\pi(v,w)\}_{(v,w)\in E_B}\right)$$

of Unique Games consists of a biregular bipartite graph $B(V_B \cup W_B, E_B)$ and a set $[R]$ of labels. For each edge $(v,w) \in E_B$ there is a constraint specified by a permutation $\pi(v,w) : [R] \to [R]$. The goal is to find a labeling $\ell : V_B \cup W_B \to [R]$ of the vertices such that as many edges as possible are satisfied, where an edge $e = (v,w)$ is said to be satisfied if $\ell(v) = \pi(v,w)(\ell(w))$.

Definition 3.2. Given a Unique Games instance

$$\mathcal{L}\left(B(V_B \cup W_B, E_B), [R], \{\pi(v,w)\}_{(v,w)\in E_B}\right),$$

let $\mathsf{Opt}(\mathcal{L})$ denote the maximum fraction of simultaneously-satisfied edges of $\mathcal{L}$ by any labeling, i.e.,

$$\mathsf{Opt}(\mathcal{L}) := \frac{1}{|E|} \max_{\ell : V_B \cup W_B \to [R]} |\{e \in E : \ell \text{ satisfies } e\}|.$$

Conjecture 3.3 (The Unique Games Conjecture [13])**.** *For any constants $\eta > 0$, there is $R = R(\eta)$ such that, for a Unique Games instance $\mathcal{L}$ with label set $[R]$, it is* NP*-hard to distinguish between the following cases.*

- $\mathsf{opt}(\mathcal{L}) \geq 1 - \eta$.

- $\mathsf{opt}(\mathcal{L}) \leq \eta$.

To show the optimal hardness result for Vertex Cover, Khot and Regev [14] introduced the following seemingly stronger conjecture, and proved that it is in fact equivalent to the original Unique Games Conjecture.

Conjecture 3.4 (Khot and Regev [14]). *For any constants $\eta > 0$, there is $R = R(\eta)$ such that, for a Unique Games instance $\mathcal{L}$ with label set $[R]$, it is NP-hard to distinguish between the following cases.*

- *There is a set $W' \subseteq W_B$ such that $|W'| \geq (1-\eta)|W_B|$ and a labeling $\ell : V_B \cup W_B \to [R]$ that satisfies every edge (v,w) for $v \in V_B$ and $w \in W'$.*
- $\mathsf{opt}(\mathcal{L}) \leq \eta$.

We describe the reduction from Unique Games. It is parametrized by an integer k and $\varepsilon \in (0, 1/(k+1))$ as in the statement of Theorem 1.1 and another parameter R that will be chosen later. Note that k and ε determine the correlated space (Ω^k, μ') as in the previous section.

Given an instance $\mathcal{L}$ of Unique Games, we assign to each vertex $w \in W_B$ the hypercube Ω^R_w. Formally, $V_G = V_H := W_B \times \Omega^R$. The weight of each vertex (w,x) is the weight of x in Ω^R divided by $|W_B|$, so that the sum of the weights is again 1.

For a permutation $\sigma : [R] \to [R]$, let $x \circ \sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(R)})$. The weighted hyperedges of H are again defined by the following procedure to sample k vertices $(w^1,x^1), \ldots, (w^k,x^k)$.

- Sample $v \in V_B$ uniformly at random.
- Sample k vertices $w^1, \ldots, w^k \in W_B$ i. i. d. from neighbors of v.
- Sample $x^1, \ldots, x^k \in \Omega^R$ from the dictatorship distribution.
- Return the hyperedge $((w^1, x^1 \circ \pi(v,w^1)), \ldots, (w^k, x^k \circ \pi(v,w^k)))$.

For each non-defective hyperedge $((w^1,x^1), \ldots, (w^k,x^k))$, we add k edges

$$((w^1,x^1),(w^2,x^2)), \ldots, ((w^k,x^k),(w^1,x^1))$$

to G.

Completeness. Suppose there exists a labeling ℓ and a subset $W' \subseteq W_B$ with $|W'| \geq (1-\eta)|W_B|$ such that ℓ satisfies every edge incident on W'. For $1 \leq i \leq k$, let

$$V_i := \bigcup_{w \in W'} \{(w,x) : x_{\ell(w)} = i\}$$

and $V_0 := V_G \setminus (\bigcup_{i=1}^k V_i)$. Note that for $i \in [k]$, $\mu(V_i) \geq (1-\eta)(1-\varepsilon)/k$. Let G' be the induced subgraph on $V_G \setminus V_0$. For any edge $((w^1,x^1),(w^2,x^2)) \in E_{G'}$, we know $w^1, w^2 \in W'$ and they share a neighbor $v \in V_B$. By the property of our dictatorship test, for each $1 \leq j \leq R$,

$$a := (x^1)_{\pi(v,w^1)^{-1}(j)} \quad \text{and} \quad b := (x^2)_{\pi(v,w^2)^{-1}(j)}$$

satisfy that at least one of them is zero or $b = [a+1]_k$. Therefore, if $(w^1,x^1),(w^2,x^2) \notin V_0$, which implies

$$(x^1)_{\pi(v,w^1)^{-1}(\ell(v))} = (x^1)_{\ell(w^1)}, \quad (x^2)_{\pi(v,w^2)^{-1}(\ell(v))} = (x^2)_{\ell(w^2)}$$

are nonzero, we can conlude that $(w^1,x^1) \in V_i$ and $(w^2,x^2) \in V_{[i+1]_k}$ for some $1 \leq i \leq k$.

Soundness. The soundness anlaysis is standard and closely follows Bansal and Khot [2]. Suppose $A \subseteq V_H$ of measure at least β such that it is independent (i.e., does not contain any non-defective hyperedge). We will show that the instance $\mathcal{L}$ of Unique Games admits a good labeling. Its contrapositive shows that if $\mathcal{L}$ does not admit a good labeling, any subset of measure at least β contains a non-defect hyperedge and the corresponding k-cycle, proving Theorem 1.1.

Let $A_w = \Omega_w^R \cap A$ be the vertices of A that lie in Ω_w^R for $w \in W_B$. Let $f_w : \Omega^R \to \{0,1\}$ be the indicator function of A_w. Define $f_v : \Omega^R \to [0,1]$ for each $v \in V_B$ to be

$$f_v(x) = \mathbb{E}_{w \in N(v)}[f_w(x \circ \pi(v,w))]$$

where $N(v)$ is the set of neighbors of v. Since B is biregular, $\mathbb{E}_{v,x}[f_v(x)] \geq \beta$. By an averaging argument, at least $\beta/2$ fraction of vertices in V_B satisfy $\mathbb{E}_x[f_v(x)] \geq \beta/2$. Call such vertices *good*.

Since A is an independent set, for any $v \in V$ and its k neighbors $w^1, \ldots, w^k$, we have

$$\mathbb{E}_{x^1,\ldots,x^k}\Big[\prod_{i=1}^{k} f_{w^i}(x^i \circ \pi(v,w^i))\Big] \leq k^2 \varepsilon^{2R}.$$

Averaging over all k-tuples $w^1, \ldots, w^k$ of neighbors of v, we have

$$\mathbb{E}_{x^1,\ldots,x^k}\Big[\prod_{i=1}^{k} f_v(x^i)\Big] = \mathbb{E}_{x^1,\ldots,x^k}\mathbb{E}_{w^1,\ldots,w^k \in N(v)}\Big[\prod_{i=1}^{k} f_{w^i}(x^i \circ \pi(v,w^i))\Big] \leq k^2 \varepsilon^{2R}.$$

Applying Theorem 2.2 (take R large enough to make sure that $k^2\varepsilon^{2R} \ll \delta$), there exist τ and d such that f_v has a coordinate j with $\mathsf{Inf}_j^{\leq d}(f_v) \geq \tau$. Set $\ell(v) = j$. Since

$$\begin{aligned}\mathsf{Inf}_j^{\leq d}(f_v) &= \sum_{\alpha_j \neq 0, |\alpha| \leq d} \hat{f}_v(\alpha)^2 = \sum_{\alpha_j \neq 0, |\alpha| \leq d} \big(\mathbb{E}_w[\hat{f}_w(\pi(v,w)^{-1}(\alpha))]^2\big) \\ &\leq \sum_{\alpha_j \neq 0, |\alpha| \leq d} \mathbb{E}_w\big[\hat{f}_w(\pi(v,w)^{-1}(\alpha))^2\big] = \mathbb{E}_w\big[\mathsf{Inf}_{\pi(v,w)^{-1}(j)}^{\leq d}(f_w)\big],\end{aligned}$$

at least $\tau/2$ fraction of v's neighbors satisfy $\mathsf{Inf}_{\pi(v,w)^{-1}(j)}^{\leq d}(f_w) \geq \tau/2$. There are at most $2d/\tau$ coordinates with degree-d influence at most $\tau/2$, and $\ell(w)$ is chosen uniformly among those coordinates (if there is none, set it arbitrarily). The above probabilistic strategy satisfies at least $(\beta/2)(\tau/2)(\tau/2d)$ fraction of all edges. By taking large R, η can be made less than this quantity, implying that if a Unique Games instance has value at most η, then the resulting H cannot have an independent set of measure at least β, which is equivalent to saying that every subset of V_G of measure at least β contains a k-cycle. Taking $\beta \leftarrow \varepsilon$ proves Theorem 1.1.

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